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# Properties of certain p-valently convex functions

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**Abstract**

*A subclass $C_p(\lambda,\mu)$ $(p\in\mathbb{N}, 0<\lambda<1, -\lambda\leqq\mu<1)$ of p-valently convex functions in the open unit disk $\mathbb{U}$ is introduced. The object of the present paper is to discuss some interesting properties of functions belonging to the class $C_p(\lambda,\mu)$.*

## 1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in\mathbb{N}=\{1,2,3,\cdots\})$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)$ in $\mathcal{A}_p$ is said to be p-valently convex of order $\alpha$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>p\alpha \qquad (z\in\mathbb{U})$$

for some $\alpha\,(0\leqq\alpha<1)$. We denote by $\mathcal{K}_p(\alpha)$ the subclass of $\mathcal{A}_p$ consisting of functions which are p-valently convex of order $\alpha$ in $\mathbb{U}$. In particular, we denote by $\mathcal{K}_1(0)=\mathcal{K}$.

A function $f(z)\in\mathcal{A}_1$ is said to be uniformly convex in $\mathbb{U}$ if $f(z)$ is in the class $\mathcal{K}$ and has the property that the image arc $f(\gamma)$ is convex for every circular are $\gamma$ contained in $\mathbb{U}$ with center at $t\in\mathbb{U}$. We also denote by $\mathcal{UK}$ the subclass of $\mathcal{A}_1$ consisting of all uniformly convex functions in $\mathbb{U}$. Goodman [2] has introduced the class $\mathcal{UK}$ and given that $f(z)\in\mathcal{A}_1$ belongs to the class $\mathcal{UK}$ if and only if

$$\mathrm{Re}\left\{1+(z-t)\frac{f''(z)}{f'(z)}\right\}\geqq 0 \qquad ((z,t)\in\mathbb{U}\times\mathbb{U}).$$

Ma and Minda [3] and Rønning [5] have showed a more applicable characterization for $\mathcal{UK}$. We state this as

**Theorem A.** *Let $f(z)\in\mathcal{A}_1$. Then $f(z)\in\mathcal{UK}$ if and only if*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\mathbb{U}).$$

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In view of Theorem A, Owa [4] considered a subclass $\mathcal{UK}(\mu)\,(-1 < \mu < 1)$ of $\mathcal{A}_1$. A function $f(z) \in \mathcal{A}_1$ is said to be a member of the class $\mathcal{UK}(\mu)\,(-1 < \mu < 1)$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \mu > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}).$$

In the present paper we investigate the following subclass of $\mathcal{A}_p$.

**Definition.** A function $f(z) \in \mathcal{A}_p$ is said to be a member of the class $\mathcal{C}_p(\lambda, \mu)$ if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - p\mu > \lambda\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \qquad (z \in \mathbb{U}) \tag{1}$$

for some $\lambda\,(0 < \lambda < 1)$ and $\mu\,(-\lambda \leqq \mu < 1)$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \leqq |z|$ and $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In proving our results, we need the following lemmas.

**Lemma 1.1.** *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \prec g(z)$$

*and* $g(z) \in \mathcal{K}$. *Then* $|a_n| \leqq 1\,(n = 1, 2, 3, \cdots)$.

We note that Lemma 1.1 can be seen in [1].

**Lemma 1.2.** *A function $f(z)$ in $\mathcal{A}_p$ belongs to the class $\mathcal{K}_p(\alpha)\,(0 \leqq \alpha < 1)$ if*

$$\sum_{n=1}^{\infty} (p+n)\{n + p(1-\alpha)\}|a_{p+n}| \leqq p^2(1-\alpha). \tag{2}$$

*Proof.* If the inequality (2) holds true, then we have that

$$\begin{aligned} \left|1 + \frac{zf''(z)}{f'(z)} - p\right| &= \left|\frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n}\right| \\ &\leqq \frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p - \sum_{n=1}^{\infty}(p+n)|a_{p+n}|} \leqq p(1-\alpha) \end{aligned} \tag{3}$$

for $z \in \mathbb{U}$. From (3), we easily seen that $f(z) \in \mathcal{K}_p(\alpha)$.

## 2 Subordination properties

Our first result for properties of functions $f(z) \in \mathcal{A}_p$ is contained in

**Theorem 2.1.** *A function $f(z) \in C_p(\lambda, \mu)$ if and only if*

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z)$$

*with*

$$h(z) = p + \frac{p(1-\mu)}{2\sin^2\beta}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{\frac{2\beta}{\pi}} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{\frac{2\beta}{\pi}} - 2\right\} \qquad (\beta = \arccos\lambda). \tag{4}$$

*Proof.* Let $1 + \dfrac{zf''(z)}{f'(z)} = w$ and $w = u + iv$. Then the inequality (1) can be written as

$$u - p\mu > \lambda\sqrt{(u-p)^2 + v^2}. \tag{5}$$

By computing, we find that the inequality (5) is equivalent to

$$\left(u + \frac{p(\lambda^2-\mu)}{1-\lambda^2}\right)^2 - \frac{\lambda^2}{1-\lambda^2}v^2 > \left(\frac{p\lambda(1-\mu)}{1-\lambda^2}\right)^2 \tag{6}$$

and

$$u > \frac{p(\lambda+\mu)}{1+\lambda}. \tag{7}$$

Thus the domain of the values of $1 + \dfrac{zf''(z)}{f'(z)}$ for $z \in \mathbb{U}$ is

$$\mathbb{D} = \{w = u + iv : u \text{ and } v \text{ satisfy (6) with (7)}\}.$$

In order to prove our theorem, it suffices to show that the function $h(z)$ given by (4) maps $\mathbb{U}$ conformally onto the domain $\mathbb{D}$.

Consider the transformations

$$w_1 = \frac{1-\lambda^2}{p(1-\mu)}w + \frac{\lambda^2-\mu}{1-\mu}$$

and

$$t = \frac{1}{2}\left(w_2^{\frac{\pi}{\beta}} + w_2^{-\frac{\pi}{\beta}}\right),$$

where $\beta = \arccos\lambda$ and $w_2 = w_1 + \sqrt{w_1^2 - 1}$ is the inverse function of

$$w_1 = \frac{w_2 + \frac{1}{w_2}}{2}.$$

It is easy to verify that composite function $t = t(w)$ maps $\mathbb{D}^+$ defined by

$$\mathbb{D}^+ = \{w = u + iv : u \text{ and } v \text{ satisfy (6) with (7) and } v > 0\}$$

conformally onto the upper half plane $\mathrm{Im}(t) > 0$ so that $w = p$ corresponds to $t = 1$ and $w = \dfrac{p(\lambda+\mu)}{1+\lambda}$ to $t = -1$. With the help of the symmetry principle, this function $t = t(w)$ maps $\mathbb{D}$ conformally onto the domain

$$\mathbb{G} = \{t : |\arg(t+1)| < \pi\}.$$

Since

$$t = 2\left(\frac{1+z}{1-z}\right)^2 - 1$$

maps $\mathbb{U}$ onto $\mathbb{G}$, we see thtat

$$\begin{aligned} w &= p + \frac{p(1-\mu)}{2(1-\lambda^2)}\{(t+\sqrt{t^2-1})^{\frac{\beta}{\pi}} + (t+\sqrt{t^2-1})^{-\frac{\beta}{\pi}} - 2\} \\ &= p + \frac{p(1-\mu)}{2\sin^2\beta}\left\{\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2\frac{\beta}{\pi}} + \left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^{2\frac{\beta}{\pi}} - 2\right\} \\ &= h(z) \end{aligned}$$

maps $\mathbb{U}$ onto $\mathbb{D}$ with $h(0) = p$. Hence the proof of the theorem is completed.

□

Theorem 2.1 gives the following corollaries.

**Corollary 2.1.** *If $f(z) \in \mathcal{C}_p(\lambda,\mu)$, then $f(z) \in \mathcal{K}_p\left(\frac{\lambda+\mu}{1+\lambda}\right)$ and the order $\frac{\lambda+\mu}{1+\lambda}$ is sharp with the extremal function*

$$f_0(z) = p\int_0^z \left(t_2^{p-1}\exp\int_0^{t_2}\frac{h(t_1)-p}{t_1}\,dt_1\right)dt_2\,, \tag{8}$$

*where $h(z)$ is given by* (4).

*Proof.* Using (7) in the proof of Theorem 2.1 and noting that

$$\mathrm{Re}\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) = \mathrm{Re}(h(z)) \to p\frac{\lambda+\mu}{1+\lambda}$$

as $z = \mathrm{Re}(z) \to -1$, we have the corollary.

□

**Corollary 2.2.** *If $f(z) \in \mathbb{C}_p(\lambda,\mu)$ and $-\lambda < \mu < \lambda < 1$, then*

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \arctan\left(\frac{1-\mu}{\sqrt{\lambda^2-\mu^2}}\right) \qquad (z \in \mathbb{U}). \tag{9}$$

*The bound in* (9) *is sharp with the extremal function $f_0(z)$ given by* (8).

*Proof.* Let the function $h(z)$ be defined by (4). Then $h(\mathbb{U}) = \mathbb{D}$ and an easy calculation yields that

$$\min\{\theta : |\arg(h(z))| < \theta \ (z \in \mathbb{U})\} = \arctan\left(\frac{1-\mu}{\sqrt{\lambda^2-\mu^2}}\right)$$

for $-\lambda < \mu < \lambda < 1$. Therefore the corollary follows immediately from Theorem 2.1.

□

Next we derive

**Theorem 2.2.** *Let $f(z) \in \mathcal{C}_p(\lambda, \mu)$ and $h(z)$ be defined by* (4). *Then*

$$\frac{f'(z)}{pz^{p-1}} \prec \exp \int_0^z \frac{h(t)-p}{t}\, dt \tag{10}$$

*and*

$$\left|\frac{f'(z)}{pz^{p-1}}\right| < \exp \int_0^1 \frac{h(\rho)-p}{\rho}\, d\rho \qquad (z \in \mathbb{U}). \tag{11}$$

*The bound in* (11) *is sharp with the extremal function $f_0(z)$ given by* (8).

*Proof.* Since the function $h(z) - p$ is univalent and starlike (with respect to the origin), by Theorem 2.1 and the result due to Suffridge [6, Theorem 3], we have

$$\log\left(\frac{f'(z)}{pz^{p-1}}\right) = \int_0^z \left(\frac{f''(t)}{f'(t)} - \frac{p-1}{t}\right) dt \prec \int_0^z \frac{h(t)-p}{t}\, dt, \tag{12}$$

which implies the subordination (10).

Furthermore, noting that the univalent function $h(z)$ maps the disk $|z| < \rho\,(0 < \rho \leqq 1)$ onto the domain which is convex and symmetric with respect to the real axis, we deduce that

$$\operatorname{Re} \int_0^z \frac{h(t)-p}{t}\, dt = \int_0^1 \frac{\operatorname{Re}\{h(\rho z)-p\}}{\rho}\, d\rho < \int_0^1 \frac{h(\rho)-p}{\rho}\, d\rho \tag{13}$$

for $z \in \mathbb{U}$. Thus the inequality (11) follows from (12) and (13).

□

**Remark.** If we let $\beta = \dfrac{\pi}{4}$ and $x = \left(\dfrac{1+\sqrt{\rho}}{1-\sqrt{\rho}}\right)^{\frac{1}{2}}$ $(0 \leqq \rho < 1)$, then

$$\int_0^1 \left\{\left(\frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}\right)^{2\frac{\beta}{\pi}} + \left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{2\frac{\beta}{\pi}} - 2\right\} \frac{d\rho}{\rho}$$

$$= 8\int_1^{+\infty} \left(\frac{x}{x^2+1} - \frac{1}{x+1}\right) dx = 4\log 2.$$

Thus, as the special case of Theorem 2.2, we have that if $f(z) \in \mathcal{C}_p(\frac{1}{\sqrt{2}}, \mu)\ (-\frac{1}{\sqrt{2}} \leqq \mu < 1)$, then

$$\left|\frac{f'(z)}{pz^{p-1}}\right| < 16^{p(1-\mu)} \qquad (z \in \mathbb{U}),$$

and the result is sharp.

# 3 Coefficient inequalities

**Theorem 3.1.** *If*

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

*belongs to $\mathcal{C}_p(\lambda, \mu)$, then*

$$|a_{p+1}| \leqq \frac{8p^2(1-\mu)}{p+1}\left(\frac{\beta}{\pi \sin\beta}\right)^2 \qquad (\beta = \arccos\lambda). \tag{14}$$

*The result is sharp.*

*Proof.* It can be easily verified that

$$1 + \frac{zf''(z)}{f'(z)} = p + \left(1 + \frac{1}{p}\right) a_{p+1} z + \cdots \tag{15}$$

and

$$\begin{aligned} h(z) &= p + \frac{p(1-\mu)}{2\sin^2\beta}\left(\frac{8\beta}{\pi} + \frac{8\beta}{\pi}\left(\frac{2\beta}{\pi} - 1\right)\right) z + \cdots \\ &= p + 8p(1-\mu)\left(\frac{\beta}{\pi\sin\beta}\right)^2 z + \cdots, \end{aligned} \tag{16}$$

where $h(z)$ is given by (4). Since

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \in \mathcal{C}_p(\lambda, \mu),$$

it follows from (15), (16) and Theorem 2.1 that

$$\begin{aligned} \frac{\pi^2}{8p(1-\mu)}\left(\frac{\sin\beta}{\beta}\right)^2\left(1 + \frac{zf''(z)}{f'(z)} - p\right) &= \frac{p+1}{8p^2(1-\mu)}\left(\frac{\pi\sin\beta}{\beta}\right)^2 a_{p+1} z + \cdots \\ &\prec \frac{\pi^2}{8p(1-\mu)}\left(\frac{\sin\beta}{\beta}\right)^2 (h(z) - p). \end{aligned}$$

In view of

$$\frac{\pi^2}{8p(1-\mu)}\left(\frac{\sin\beta}{\beta}\right)^2 (h(z) - p) \in \mathcal{K},$$

we get (14) by using Lemma 1.1. Also the bound in (14) ia sharp for the function $f_0(z)$ given by (8).

□

Next we see

**Theorem 3.2.** *If the function*

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

*belonging to the class $\mathcal{A}_p$ satisfies*

$$\sum_{n=1}^{\infty}(p+n)\{n(1+\lambda)+p(1-\mu)\}|a_{p+n}| \leqq p^2(1-\mu), \tag{17}$$

*then $f(z)$ belongs to the class $\mathcal{C}_p(\lambda,\mu)$.*

*Proof.* Applying the inequality (17), we deduce that

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - p\mu - \lambda\left|1+\frac{zf''(z)}{f'(z)} - p\right| \\
&\geqq p(1-\mu) - (1+\lambda)\left|1+\frac{zf''(z)}{f'(z)} - p\right| \\
&= p(1-\mu) - (1+\lambda)\left|\frac{\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p+\sum_{n=1}^{\infty}(p+n)a_{p+n}z^n}\right| \\
&\geqq p(1-\mu) - (1+\lambda)\left(\frac{\sum_{n=1}^{\infty} n(p+n)|a_{p+n}|}{p-\sum_{n=1}^{\infty}(p+n)|a_{p+n}|}\right) \\
&\geqq 0,
\end{aligned}$$

which shows that $f(z) \in \mathcal{C}_p(\lambda,\mu)$.

□

By using Theorem 3.2 and Corollary 2.1, we easily have

**Corollary 3.1.** *Let*

$$f(z) = z^p + \sum_{n=1}^{\infty}(-1)^{n+1}|a_{p+n}|z^{p+n}$$

*be in the class $\mathcal{A}_p$. Then $f(z)$ belongs to the class $\mathcal{C}_p(\lambda,\mu)$ if and only if $f(z) \in \mathcal{K}_p\left(\frac{\lambda+\mu}{1+\lambda}\right)$.*

Finally, we derive

**Theorem 3.3.** *A function $f(z) = z^p + a_{p+n}z^{p+n}$ $(n \in \mathbb{N})$ is in the class $\mathcal{C}_p(\lambda,\mu)$ if and only if*

$$|a_{p+n}| \leqq \frac{p^2(1-\mu)}{(p+n)\{n(1+\lambda)+p(1-\mu)\}}. \tag{18}$$

*Proof.* In view of Theorem 3.2, it suffices to show the only if part. Let us suppose that $f(z) \in \mathcal{C}_p(\lambda,\mu)$. Then

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - p\mu - \lambda\left|1+\frac{zf''(z)}{f'(z)} - p\right|$$

$$= p(1-\mu) + \mathrm{Re}\left(\frac{n(p+n)a_{p+n}z^n}{p+(p+n)a_{p+n}z^n}\right) - \lambda\left|\frac{n(p+n)a_{p+n}z^n}{p+(p+n)a_{p+n}z^n}\right| > 0. \tag{19}$$

Writing $a_{p+n} = |a_{p+n}|e^{i\theta}$ ($\neq 0$) and letting $z \to e^{i\frac{\pi-\theta}{n}}$ ($z \in \mathbb{U}$), we have $a_{p+n}z^n \to -|a_{p+n}|$ and it follows from (19) that

$$p(1-\mu) - (1+\lambda)\frac{n(p+n)|a_{p+n}|}{p-(p+n)|a_{p+n}|} \geqq 0,$$

which implies the inequality (18).

□

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